Geometric optics and the Cauchy problem for nonlinear Schrödinger equation

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Geometric optics for Schrödinger equation

Schrödinger equation in a semi-classical regime ($\varepsilon \ll 1$):

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon = 0 \quad ; \quad \psi^\varepsilon(0,x) = a_0(x)e^{i\phi_0(x)/\varepsilon},$$

where $\psi^\varepsilon : \mathbb{R}_t \times \mathbb{R}^d_x \to \mathbb{C}$.

WKB: seek $\psi^\varepsilon(t,x) \underset{\varepsilon\to 0}{\sim} a(t,x)e^{i\phi(t,x)/\varepsilon}$.

$\mathcal{O}(\varepsilon^0)$: eikonal equation, $\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 = 0 \quad ; \quad \phi_{|t=0} = \phi_0$.

$\mathcal{O}(\varepsilon^1)$: transport equation, $\partial_t a + \nabla\phi\cdot\nabla a + \frac{1}{2}a\Delta\phi = 0 \quad ; \quad a_{|t=0} = a_0$.

Example

If $\phi_0(x) = \kappa\cdot x$: $\phi(t,x) = \kappa\cdot x - \frac{|\kappa|^2}{2}t \rightsquigarrow$ global.

If $\phi_0(x) = a|x|^2$: $\phi(t,x) = \frac{2a}{at+2}|x|^2 \rightsquigarrow$ focusing at $t = -2/a$.

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Nonlinear geometric optics

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon = |\psi^\varepsilon|^{2\sigma}\psi^\varepsilon \quad ; \quad \psi^\varepsilon(0,x) = \varepsilon^J a_0(x)e^{i\phi_0(x)/\varepsilon}.$$

Defocusing nonlinearity: "no blow-up".

Equivalently:

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon = \varepsilon^\alpha|\psi^\varepsilon|^{2\sigma}\psi^\varepsilon \quad ; \quad \psi^\varepsilon(0,x) = a_0(x)e^{i\phi_0(x)/\varepsilon}.$$

As in the linear case, seek

$$\psi^\varepsilon(t,x) \underset{\varepsilon\to 0}{\sim} a(t,x)e^{i\phi(t,x)/\varepsilon}$$

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Notion of criticality

Plug the *ansatz* into the equation:

$$\mathcal{O}\left(\varepsilon^0\right) : \partial_t\phi + \frac{1}{2}|\nabla\phi|^2 = \begin{cases} 0 & \text{if } \alpha > 0 \\ -|a|^{2\sigma} & \text{if } \alpha = 0 \end{cases} \quad ; \quad \phi_{|t=0} = \phi_0.$$

$$\mathcal{O}\left(\varepsilon^1\right) : \partial_t a + \nabla\phi\cdot\nabla a + \frac{1}{2}a\Delta\phi = \begin{cases} 0 & \text{if } \alpha > 1 \\ -i|a|^{2\sigma}a & \text{if } \alpha = 1 \\ ?? & \text{if } \alpha < 1 \end{cases} ; \; a_{|t=0} = a_0.$$

Critical values: $\alpha_c = 1$: "first" nonlinear effects (transport equation).
$\alpha_c' = 0$: strongest nonlinear effects (eikonal equation).

Remark

If $\alpha \geqslant 1$: same eikonal equation as in the linear case.
If $\alpha = 0$ and $\phi_0 = 0$, we have $\partial_t\phi_{|t=0} \neq 0$ (unless $a_0 = 0$).

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A notion of well-posedness

$$(*) \qquad \partial_t u + L(\partial_x)u = F(u) \quad ; \quad u_{|t=0} = u_0.$$

Definition (From KPV01)

The Cauchy problem is well posed from H^s to H^k if, for all bounded subset $B \subset H^s$, there exist $T > 0$ and a Banach space $X_T \hookrightarrow C([0,T]; H^k)$ such that:
(1) For all $u_0 \in H^s$, (*) has a unique solution $u \in X_T$.
(2) $u_0 \in (B, \|\cdot\|_{H^s}) \mapsto u \in C([0,T]; H^k)$ is continuous.

Critical thresholds

(NLS) $$i\partial_t u + \frac{1}{2}\Delta u = |u|^{2\sigma} u \quad ; \quad u_{|t=0} = u_0.$$

Two (of the) conserved quantities:

$$M = \int_{\mathbb{R}^d} |u(t,x)|^2 dx,$$
$$E = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u(t,x)|^2 dx + \frac{1}{\sigma+1}\int_{\mathbb{R}^n} |u(t,x)|^{2\sigma+2} dx.$$

Two important invariances:

- $u(t,x) \mapsto \lambda^{1/\sigma} u\left(\lambda^2 t, \lambda x\right)$, $\lambda > 0$: $\dot{H}^{s_c}_x$-norm invariant, $s_c = \frac{d}{2} - \frac{1}{\sigma}$.
- $u(t,x) \mapsto e^{iv\cdot x - i|v|^2 t/2} u(t, x - vt)$, $v \in \mathbb{R}^d$: L^2_x-norm invariant.

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Well-posedness

- $s_c \geqslant 0$: well-posedness $H^s(\mathbb{R}^d) \to H^s(\mathbb{R}^d)$ for $s > s_c$.
  (Cazenave-Weissler 90')
- $s_c < 0$: well-posedness $H^s(\mathbb{R}^d) \to H^s(\mathbb{R}^d)$ for $s \geqslant 0$.
  (Tsutsumi 87')

Lack of well-posedness: $s > 0$

Assume $s_c > 0$: lack of well-posedness $H^s(\mathbb{R}^d) \to H^s(\mathbb{R}^d)$ for $0 < s < s_c$.

- Lebeau for the wave equation $\partial_t^2 u - \Delta u + u^p = 0$, $x \in \mathbb{R}^3$, $p \in 2\mathbb{N}+1$, $p \geqslant 7$; Séminaire Bourbaki by Guy Métivier.
- (NLS): Christ-Colliander-Tao, Burq-Gérard-Tzvetkov.

Argument: concentrated initial data, $u_0(x) = h^M a_0\left(\frac{x}{h}\right)$, $h \to 0$.
Boundedness in $H^s(\mathbb{R}^d)$: $M - s \geqslant -d/2$.
Scaling $\rightsquigarrow$ supercritical geometric optics:

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon = |\psi^\varepsilon|^{2\sigma}\psi^\varepsilon \quad ; \quad \psi^\varepsilon_{|t=0} = a_0.$$

For very short time ($t \leqslant C\varepsilon|\ln\varepsilon|^\theta$), Laplacian negligible.
Appearance of oscillations (ODE mechanism): decoherence, "norm inflation".

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Loss of regularity

Theorem (RC, T. Alazard-RC, L. Thomann)

Let $\sigma \geqslant 1$. Assume that $s_c = d/2 - 1/\sigma > 0$, and let $0 < s < s_c$. There exists a family $(u_0^h)_{0<h\leqslant 1}$ in $\mathcal{S}(\mathbb{R}^d)$ with

$$\|u_0^h\|_{H^s(\mathbb{R}^d)} \to 0 \text{ as } h \to 0,$$

a solution u^h to (NLS) and $0 < t^h \to 0$, such that:

$$\|u^h(t^h)\|_{H^k(\mathbb{R}^d)} \to +\infty \text{ as } h \to 0\,,\ \forall k > \frac{s}{1+\sigma(s_c - s)}.$$

Corollary

Let $\sigma \geqslant 1$. Assume that $s_c = d/2 - 1/\sigma > 0$, and let $0 < s < s_c$. (NLS) is not locally well-posed from H^s to H^k, for all $k > \frac{s}{1+\sigma(s_c-s)}$.

Let $s_{\rm sob} = \frac{d}{2}\frac{\sigma}{\sigma+1}$: corresponds to the embedding $H^{s_{\rm sob}}(\mathbb{R}^d) \subset L^{2\sigma+2}(\mathbb{R}^d)$.

$$\frac{s_{\rm sob}}{1+\sigma(s_c - s_{\rm sob})} = 1.$$

Corollary

Let $d \geqslant 3$ and $\sigma > \frac{2}{d-2}$. There exists a family $(u_0^h)_{0<h\leqslant 1}$ in $\mathcal{S}(\mathbb{R}^d)$ with

$$M^h + E^h \to 0 \text{ as } h \to 0,$$

a solution u^h to (NLS) and $0 < t^h \to 0$, such that:

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Remark

Analogue of the result due to G. Lebeau in the case of the wave equation.

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$$u_0^h(x) = h^{s-d/2} a_0\left(\frac{x}{h}\right), \quad h \to 0 : \text{ bounded in } H^s, \text{ but not in } H^{s^+}.$$

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$$\varepsilon = h^{\sigma(s_c - s)} \underset{h \to 0}{\longrightarrow} 0 \quad ; \quad \psi^\varepsilon(t,x) = h^{\frac{n}{2}-s} u^h\left(h^2 \varepsilon t, hx\right).$$

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Quasi-linear analysis is needed there (or analytic setting).

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$\mathbb{R}^d$: lack of wp $H^s \to H^s$, norm inflation if $s \leqslant -d/2$ [CCT].

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$\rightsquigarrow$ [CDS2]: lack of wp $H^s(\mathbb{R}^d) \to H^k(\mathbb{R}^d)$, $\forall s < 0, \forall k \in \mathbb{R}$.

Theorem (CDS2)

Suppose $d \geqslant 2$, $\sigma \in \mathbb{N}$. Let $s < -1/(2\sigma+1)$. There exists a family $(u_0^h)_{0<h\leqslant 1}$ in $\mathcal{S}(\mathbb{R}^d)$ with

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A useful lemma

Lemma

Let $d \geqslant 1$, $\beta > 0$. *For* $f \in \mathcal{S}'(\mathbb{R}^d)$, $\kappa \in \mathbb{R}^d$:

$$I^\varepsilon(f,\kappa)(x) := f\left(x\varepsilon^{(1-\beta)/2}\right) e^{i\kappa\cdot x/\varepsilon^{(1+\beta)/2}}.$$

(1) $\kappa \neq 0$: $\forall s \leqslant 0$, $\exists C = C(s,\kappa)$ *such that* $\forall f \in \mathcal{S}(\mathbb{R}^d)$,

$$\|I^\varepsilon(f,\kappa)\|^2_{H^s(\mathbb{R}^d)} \leqslant C\varepsilon^{-d(1-\beta)/2+(1+\beta)|s|}\|f\|^2_{H^m(\mathbb{R}^d)}.$$

(2) *For all* $s \leqslant 0$, $\beta < 1$ *and* $f \in L^2(\mathbb{R}^d)$,

$$\|I^\varepsilon(f,0)\|^2_{H^s(\mathbb{R}^d)} = \varepsilon^{-d(1-\beta)/2}\left(\|f\|^2_{L^2(\mathbb{R}^d)} + o(1)\right), \quad as\ \varepsilon \to 0.$$

(3) *If* $\beta > 1$, $s \leqslant 0$, *and* $f \in H^s(\mathbb{R}^d)$,

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If:

1. the zero mode is generated by nonlinear interaction of nonzero initial modes,
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Multiphase WNLGO

Linear phases: $\phi_j(t,x) = \kappa_j \cdot x - \frac{|\kappa_j|^2}{2} t.$

Nonlinear interaction $\rightsquigarrow \phi = \phi_1 - \phi_2 + \phi_3 - \cdots + \phi_{2\sigma+1}$:

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Resonance: $\omega = |\kappa|^2/2$ (otherwise: nonstationary phase).

Cubic case ($\sigma = 1$): resonances fully described by an algorithm based on the completion of rectangles (Colliander-Keel-Staffilani-Takaoka-Tao).

$\sigma \geqslant 2$: geometric insight more intricate.

Remark

If (ϕ_1, ϕ_2, ϕ_3) generates a resonance when $\sigma = 1$, then so does $(\phi_1, \phi_2, \phi_3, \dots, \phi_3)$ for $\sigma \geqslant 2$.

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The key example

Example

Set of initial phases:

$$\Phi_0 = \{\kappa_1 = (1,0,\dots,0), \kappa_2 = (1,1,0,\dots,0), \kappa_3 = (0,1,0,\dots,0)\}\,.$$

Cubic nonlinearity ($\sigma = 1$): the set of *relevant* phases is

$$\Phi = \Phi_0 \cup \{\kappa_0 = 0_{\mathbb{R}^d}\}.$$

Higher order nonlinearities ($\sigma \geqslant 2$): $0 \in \Phi$.

Amplitudes

A word of caution: the geometry of phases does not suffices for the effective appearance of a new mode.

Example

$d = 1$ (flat rectangles): nonlinear modulation of the amplitudes = phase modulation.
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General transport system

$$\partial_t a_j + \kappa_j \cdot \nabla a_j = -i \sum_{(\ell_1,\dots,\ell_{2\sigma+1})\in I_j} a_{\ell_1}\overline{a}_{\ell_2}\dots a_{\ell_{2\sigma+1}} \quad ; \quad a_{j|t=0} = \alpha_j.$$

Lemma

$\sigma \in \mathbb{N}^*$, $d \geqslant 2$. Consider the key example. There exist $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{S}(\mathbb{R}^d)$ such that if we set $\kappa_0 = 0_{\mathbb{R}^d}$,

$$\partial_t a_{0|t=0} \neq 0.$$

For instance, this is so if $\alpha_1 = \alpha_2 = \alpha_3 \neq 0$.

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From $s < -1/(2\sigma)$ to $s < -1/(2\sigma+1)$

More weakly NLGO: $J > 1$,

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon = \varepsilon^J|\psi^\varepsilon|^{2\sigma}\psi^\varepsilon \quad ; \quad \psi^\varepsilon(0,x) = \sum_{j\in J_0}\alpha_j(x)e^{i\kappa_j\cdot x/\varepsilon}.$$

$\rightsquigarrow$ $J > 1$: NL effects are negligible at leading order in $L^2 \cap L^\infty$.

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